On the Nature of Proof

Generally a proposition is 'proved' if most people agree to it. This is not always the case, since most people believed the sun orbited the earth, whereas it is the other way around - as determined by "science" (telescopic observation) rather than any egocentric belief.
There are various levels of proof in society. In law one needs evidence to prove a case; however, one can go from circumstantial evidence (say where no body is found) to much stronger DNA evidence which is almost irrefutable.

To any statement where the question is asked "Can you prove it?" the author can ask in return "Well, what sort of proof do you want?", which may be hard to answer.

On Mathematical Proof (alternatively: not everyone's cup of tea!)
Mathematics is a subject dealing with propositions which are regarded as either true or false, such as 2+2=4 or 2+2=22! The exclamation "Good heavens!" is not a proposition.

The Principle of Mathematical Induction is a logical proof taught in 1^{st} year University Mathematics. Accepting all previous n^{th} stages, including the first, if you can then prove the $(n+1)^{th}$ stage then the result is true for all n, where n is a natural number (1,2,3,…)
For example: What is the sum of all numbers from 1 to any number?
Assume the sum of the numbers from 1 to n is $s_n = n(n+1)/2$. Add the next number, which would be n+1 to get $n(n+1)/2 + (n+1) = (n^2 + n + 2n + 2)/2 = (n^2 + 3n + 2)/2 = (n+1)(n+2)/2$
So we see that the result is true where n is replaced by n+1 in the formula for s_n so that
$s_{n+1} = (n+1)(n+2)/2$.

The usual set of axioms one accepts (most of the world, including most mathematicians) are ZFC (Zermelo-Fraenkel axioms with the axiom of Choice in Set Theory) and most mathematics proceeds on accepting the 8 axioms and the Choice axiom). Look these up if interested.

Researchers in Set Theory comment upon, and work on, weaker and stronger axioms than ZFC.

Set Theory language (symbols) is the language of *Principia Mathematica.*
An example of an axiom in ZFC is the Axiom of Extension, using the usual set theory symbols

Axiom of extension in ZFC

Two sets are equal (are the same set) if they have the same elements (x and y are sets and z is an element of a set)

$$\forall x \forall y [\forall z (z \in x \Leftrightarrow z \in y) \Rightarrow x = y]$$

This reads as: for all x and for all y, such that, for all z such that z belongs to x implies z belongs to y, and vice versa, this implies x=y.

The most famous mathematical problem, which remained unsolved for centuries was "Fermat's Last Theorem" of 1637. This was finally solved by Andrew Wiles in 1995, in two papers which totalled 129 pages.
The most famous outstanding problem today is the "Riemann Hypothesis".
Others are "Goldbach's conjecture" and the "Twin Primes" problem.

Mathematics and computer development has reached the stage where theorem-proving computer programs exist for many problems.

One famous outstanding problem was the 4-colour map problem.
"It states that, given any separation of a plane into contiguous regions, producing a figure called a map, no more than four colours are required to colour the regions of the map so that no two adjacent regions have the same colour. Two regions are called adjacent if they share a common boundary that is not a corner, where corners are the points shared by three or more regions.
The four colour theorem was proved in 1976 by Kenneth Appel and Wolfgang Haken.
It was the first major theorem to be proved using a computer.
Initially, their proof was not accepted by all mathematicians because the computer-assisted proof was infeasible for a human to check by hand (Swart 1980). Since then the proof has gained wider acceptance, although doubts remain (Wilson 2014, 216–222).
To dispel remaining doubt about the Appel–Haken proof, a simpler proof using the same ideas and still relying on computers was published in 1997 by Robertson, Sanders, Seymour, and Thomas.
Additionally, in 2005, the theorem was proved by Georges Gonthier with general-purpose theorem-proving software."

A revolutionary development in mathematics, which has implications in philosophy, arose from the work of Kurt Gödel (pronounced girdle) in 1931.
To quote from the book ***"Gödel's Proof*** (Nagel and Newman)":
"People were convinced that mathematical thinking could be captured by laws of pure symbol manipulation - (all the symbols in Set Theory).
From a fixed set of axioms and a fixed set of typographical rules, one could shunt symbols around and produce new strings of symbols, called "theorems". The pinnacle of this movement was a monumental three-volume work by Bertrand Russell and Alfred North Whitehead called Principia Mathematica (PM), which came out in the years 1910-1913. Russell and Whitehead believed that they had grounded all of mathematics in pure logic, and that their work would form the solid foundation for all of mathematics forevermore."

A couple of decades later, Gödel began to doubt this noble vision. Gödel realized that, in principle, he could write down a formula of Principia Mathematica that perversely said about itself, "This formula is unprovable by the rules of Principia Mathematica."

This led him to writing two very famous papers (in German) known as the First and Second Incompleteness Theorems.
To quote a second book: ***"Gödel 's Theorem: An Incomplete Guide to its Use and Abuse"***
"The first incompleteness theorem established that on the assumption that the system of PM satisfies a property that Gödel called omega consistency (now just plain consistency-IRex), it is incomplete, meaning that there is a statement in the language of the system that can be neither proved nor disproved in the system. Such a statement is said to be undecidable in the system.
The second incompleteness theorem showed that if the system is consistent - meaning that there is no statement in the language of the system that can be both proved and disproved in the system - the consistency cannot be established within the system."

(Very bluntly - a mathematical system cannot prove its own self-consistency - IRex)

The results of Gödel are truly remarkable and shocked the mathematical world.
He was a contemporary and colleague of Einstein where both worked at the Institute of Advanced Study in Princeton.

There have been philosophical implications of Gödel's work. In the second book above, we find, under the section "Theological Applications" the extract

Although Gödel was thus not at all averse to theological reasoning, he did not attempt to draw any theological conclusions from the incompleteness theorem. However, others have invoked the incompleteness theorem in theological discussions. *Bibliography of Christianity and Mathematics*, first edition 1983, lists 13 theological articles invoking Gödel's theorem. Here are some quotations from the abstracts of these articles:

> Nonstandard models and Gödel's incompleteness theorem point the way to God's freedom to change both the structure of knowing and the objects known.
>
> Uses Gödel's theorem to indicate that physicists will never be able to formulate a theory of physical reality that is final.
>
> Stresses the importance of Gödel's theorems of incompleteness toward developing a proper perspective of the human mind as more than just a logic machine.
>
> ...theologians can be comforted in their failure to systematize revealed truth because mathematicians cannot grasp all mathematical truths in their systems, either.
>
> If mathematics were an arbitrary creation of men's minds, we can still hold to eternal mathematical truth by appealing to Gödel's incompleteness result to guarantee truths that can be discovered only by the use of reason and not by the mechanical manipulation of fixed rules—truths which imply the existence of God.
>
> It is argued by analogy from Gödel's theorem that the methodologies, tactics, and presuppositions of science cannot be based entirely upon science; in order to decide on their validity, resources from outside science must be used.

Roger Penrose, who to my mind is a genius, has discussed Gödel's work. Also quoting from the above book, we have the extract

6.2 Penrose's "Second Argument"

Roger Penrose, in his two books *The Emperor's New Mind* and *Shadows of the Mind*, has argued at length that Gödel's theorem has implications for a "science of consciousness." In *Shadows*, he presents a Gödelian argument ("Penrose's new argument," "Penrose's second argument") aiming to establish what the Lucas argument does not, that no machine can exactly represent the ability of the human mind to prove arithmetical theorems. The presentation of this argument in *Shadows* is fairly long and involved, but fortunately Penrose has set out the argument in its essentials in the electronic journal *Psyche* ([Penrose 96]):

That is the end of my presentation.

I am still ploughing through (studying) work leading to Gödel’s Theorems – Invariant Theory is my personal research interest.

It is said that there is no mathematician now that knows “all of mathematics”. The last one who did was the French mathematician Henri Poincaré at the early 1900’s.

In relation to Penrose’s ideas above, you may care to look at the chess game, displayed above – which any chess player can see, in two minutes, that what he says is true, but no computer can – related to AI (Artificial Intelligence)!

Inspector Rex